

ON THE S^1 -FIBRED NIL-BOTT TOWER

MAYUMI NAKAYAMA

ABSTRACT. We shall introduce a notion of S^1 -fibred *nilBott tower*. It is an iterated S^1 -bundles whose top space is called an S^1 -fibred nilBott manifold and the S^1 -bundle of each stage realizes a *Seifert construction*. The nilBott tower is a generalization of *real Bott tower* from the viewpoint of fibration. In this note we shall prove that any S^1 -fibred nilBott manifold is *diffeomorphic* to an infranil-manifold. According to the group extension of each stage, there are two classes of S^1 -fibred nilBott manifolds which is defined as *finite type* or *infinite type*. We discuss their properties.

1. INTRODUCTION

Let M be a closed aspherical manifold which is a top space of an iterated S^1 -bundles over a point:

$$(1.1) \quad M = M_n \rightarrow M_{n-1} \rightarrow \dots \rightarrow M_1 \rightarrow \{\text{pt}\}.$$

Suppose X is the universal covering of M and each X_i is the universal covering of M_i and put $\pi_1(M_i) = \pi_i$ ($i = 1, \dots, n-1$) and $\pi_1(M) = \pi$.

Definition 1.1. An S^1 -fibred *nilBott tower* is a sequence (1.1) which satisfies I, II and III below ($i = 1, \dots, n-1$). The top space M is said to be an S^1 -fibred *nilBott manifold (of depth n)*.

I. M_i is a fiber space over M_{i-1} with fiber S^1 .

II. For the group extension

$$(1.2) \quad 1 \rightarrow \mathbb{Z} \rightarrow \pi_i \longrightarrow \pi_{i-1} \rightarrow 1$$

associated to the fiber space (I), there is an equivariant principal bundle:

$$(1.3) \quad \mathbb{R} \rightarrow X_i \xrightarrow{p_i} X_{i-1}.$$

III. Each π_i normalizes \mathbb{R} .

Date: October 7, 2011.

2000 *Mathematics Subject Classification.* 53C55, 57S25, 51M10 (this file draft niltower).

Key words and phrases. Aspherical manifolds, Bott tower, Seifert fiber spaces, Infranil manifolds.

The purpose of this paper is to prove the following result.

Theorem 1.2. *Suppose that M is an S^1 -fibred nilBott manifold.*

- (I) *If every cocycle of $H_\phi^2(\pi_{i-1}; \mathbb{Z})$ which represents a group extension (1.2) is of finite order, then M is diffeomorphic to a Riemannian flat manifold.*
- (II) *If there exists a cocycle of $H_\phi^2(\pi_{i-1}; \mathbb{Z})$ which represents a group extension (1.2) is of infinite order, then M is diffeomorphic to an infranilmanifold. In addition, M cannot be diffeomorphic to any Riemannian flat manifold.*

CONTENTS

1. Introduction	1
2. Seifert construction	2
3. S^1 -fibred nilBott tower	3
4. 3-dimensional S^1 -fibred nilBott towers	10
4.1. 3-dimensional S^1 -fibred nilBott manifolds of finite type	10
4.2. 3-dimensional S^1 -fibred nilBott manifolds of infinite type.	10
5. Realization	13
References	19

2. SEIFERT CONSTRUCTION

We shall explain the Seifert construction briefly. It is a tool to construct a closed aspherical manifold for a given extension.

Let

$$(2.1) \quad 1 \longrightarrow \Delta \longrightarrow \pi \xrightarrow{\nu} Q \longrightarrow 1$$

be a group extension. Then there is a conjugation function $\phi : Q \rightarrow \text{Aut}(\Delta)$ defined by a section $s : Q \rightarrow \pi$ of ν . The group extension (2.1) is represented by a cocycle $f : Q \times Q \rightarrow \Delta$ for which each element $\gamma \in \pi$ is viewed as (n, α) with group law:

$$(n, \alpha)(m, \beta) = (n \cdot \phi(\alpha)(m) \cdot f(\alpha, \beta), \alpha\beta)$$

($\forall n, m \in \Delta, \forall \alpha, \beta \in Q$) (cf. [13] for example).

Suppose Δ is a torsionfree finitely generated nilpotent group. By Mal'cev's *existence* theorem, there is a (simply connected) nilpotent Lie group \mathcal{N} containing Δ as a discrete uniform subgroup. Moreover if Q acts properly discontinuously on a contractible smooth manifold W

such that the quotient space W/Q is compact, then there is a smooth map $\lambda : Q \rightarrow \text{Map}(W, \mathcal{N})$ satisfies $f = \delta^1 \lambda$:

$$(2.2) \quad f(\alpha, \beta) = (\bar{\phi}(\alpha) \circ \lambda(\beta) \circ \alpha^{-1}) \cdot \lambda(\alpha) \cdot \lambda(\alpha\beta)^{-1} \quad (\alpha, \beta \in Q)$$

here $\bar{\phi} : Q \rightarrow \mathcal{N}$ is the extension of ϕ . And an action of π on $\mathcal{N} \times W$ is obtain by

$$(2.3) \quad (n, \alpha)(x, w) = (n \cdot \bar{\phi}(\alpha)(x) \cdot \lambda(\alpha)(\alpha w), \alpha w).$$

This action $(\pi, \mathcal{N} \times W)$ is said to be a Seifert construction. (See [5] for details.)

Taking a finite group F and $\{pt\}$ as Q and W above;

$$(2.4) \quad 1 \longrightarrow \Delta \longrightarrow \pi \xrightarrow{\nu} F \longrightarrow 1,$$

we may put \mathcal{N} as $\text{Map}(W, \mathcal{N})$ before respectively. Let \mathcal{K} be a maximal compact subgroup of $\text{Aut}(\mathcal{N})$. The group $E(\mathcal{N}) = \mathcal{N} \rtimes \mathcal{K}$ is said to be the euclidean group of \mathcal{N} . Then there is a discrete faithful representation $\rho : \pi \rightarrow E(\mathcal{N})$ which is defined by

$$(2.5) \quad \rho((n, \alpha)) = (n \cdot \chi(\alpha), \mu(\chi(\alpha)^{-1}) \circ \bar{\phi}(\alpha)) \quad (n \in \Delta, \alpha \in F),$$

where $\chi : F \rightarrow \mathcal{N}$ is a map such that $f = \delta^1 \chi$:

$$(2.6) \quad f(\alpha, \beta) = \bar{\phi}(\alpha)(\chi(\beta)) \cdot \chi(\alpha) \cdot \chi(\alpha\beta)^{-1} \quad (\alpha, \beta \in Q).$$

(See [13].) Note that the action $(\rho(\pi), \mathcal{N})$ is a Seifert construction and $\mathcal{N}/\rho(\pi)$ is an infranilmanifold (cf. [5] or [13]).

3. S^1 -FIBRED NILBOTT TOWER

This section gives the proof Theorem 1.2.

Suppose that

$$(3.1) \quad M = M_n \xrightarrow{S^1} M_{n-1} \xrightarrow{S^1} \dots \xrightarrow{S^1} M_1 \xrightarrow{S^1} \{pt\}$$

is an S^1 -fibred nilBott tower. By the definition, there is a group extension of the fiber space;

$$(3.2) \quad 1 \rightarrow \mathbb{Z} \rightarrow \pi_i \rightarrow \pi_{i-1} \rightarrow 1$$

for any i . The conjugate by each element of π_i defines a homomorphism $\phi : \pi_{i-1} \rightarrow \text{Aut}(\mathbb{Z}) = \{\pm 1\}$. With this action, \mathbb{Z} is a π_{i-1} -module so that the group cohomology $H_\phi^i(\pi_{i-1}, \mathbb{Z})$ is defined. Then the above group extension (3.2) represents a 2-cocycle in $H_\phi^2(\pi_{i-1}, \mathbb{Z})$, (cf. [13]).

Proof. Given a group extension (3.2), we suppose by induction that there exists a torsionfree finitely generated nilpotent normal subgroup Δ_{i-1} of finite index in π_{i-1} such that the induced extension $\tilde{\Delta}_i$ is a central extension:

$$(3.3) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \longrightarrow & \pi_{i-1} \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_i & \longrightarrow & \Delta_{i-1} \longrightarrow 1. \end{array}$$

It is easy to see that $\tilde{\Delta}_i$ is a torsionfree finitely generated normal nilpotent subgroup of finite index in π_i . Then π_i is a virtually nilpotent subgroup, i.e. $1 \rightarrow \tilde{\Delta}_i \rightarrow \pi_i \rightarrow F_i \rightarrow 1$ where $F_i = \pi_i/\tilde{\Delta}_i$ is a finite group. Let \tilde{N}_i, N_{i-1} be a nilpotent Lie group containing $\tilde{\Delta}_i, \Delta_{i-1}$ as a discrete cocompact subgroup respectively. Let $A(\tilde{N}_i) = \tilde{N}_i \rtimes \text{Aut}(\tilde{N}_i)$ be the affine group. If \tilde{K}_i is a maximal compact subgroup of $\text{Aut}(\tilde{N}_i)$, then the subgroup $E(\tilde{N}_i) = \tilde{N}_i \rtimes \tilde{K}_i$ is the euclidean group of \tilde{N}_i . Then there exists a faithful homomorphism (see (2.5)):

$$(3.4) \quad \rho_i : \pi_i \longrightarrow E(\tilde{N}_i)$$

for which $\rho_i|_{\tilde{\Delta}_i} = \text{id}$ and the quotient $\tilde{N}_i/\rho_i(\pi_i)$ is an infranilmanifold. The explicit formula is given by the following

$$(3.5) \quad \rho_i((n, \alpha)) = (n \cdot \chi(\alpha), \mu(\chi(\alpha)^{-1}) \circ \bar{\phi}(\alpha))$$

for $n \in \tilde{\Delta}_i, \alpha \in F$ where $\chi : F \rightarrow \tilde{\Delta}_i, \bar{\phi} : F \rightarrow \text{Aut}(\tilde{N}_i)$. As $\tilde{\Delta}_i \leq \tilde{N}_i$, there is a 1-dimensional vector space \mathbb{R} containing \mathbb{Z} as a discrete uniform subgroup which has a central group extension:

$$1 \rightarrow \mathbb{R} \rightarrow \tilde{N}_i \rightarrow N_{i-1} \rightarrow 1$$

where $N_{i-1} = \tilde{N}_i/\mathbb{R}$ is a simply connected nilpotent Lie group. As $\mathbb{Z} \leq \mathbb{R} \cap \tilde{\Delta}_i$ is discrete cocompact in \mathbb{R} and $\mathbb{R} \cap \tilde{\Delta}_i/\mathbb{Z} \rightarrow \tilde{\Delta}_i/\mathbb{Z} \cong \Delta_{i-1}$ is an inclusion, noting that Δ_{i-1} is torsionfree, it follows that $\mathbb{R} \cap \tilde{\Delta}_i = \mathbb{Z}$. We obtain the commutative diagram in which the vertical maps are inclusions:

$$(3.6) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_i & \longrightarrow & \Delta_{i-1} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{R} & \longrightarrow & \tilde{N}_i & \longrightarrow & N_{i-1} \longrightarrow 1. \end{array}$$

On the other hand, (3.4) induces the following group extension.:

$$(3.7) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \xrightarrow{p_i} & \pi_{i-1} \longrightarrow 1 \\ & & \parallel & & \rho_i \downarrow & & \hat{\rho}_i \downarrow \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \rho_i(\pi_i) & \longrightarrow & \hat{\rho}_i(\pi_{i-1}) \longrightarrow 1. \end{array}$$

Since $\tilde{\Delta}_i$ centralizes \mathbb{Z} , \tilde{N}_i centralizes \mathbf{R} . So $\hat{\rho}_i$ is a homomorphism from π_{i-1} into $E(N_{i-1})$. The explicit formula is given by the following:

$$(3.8) \quad \hat{\rho}_i((\bar{n}, \alpha)) = (\bar{n} \cdot \bar{\chi}(\alpha), \mu(\bar{\chi}(\alpha)^{-1}) \circ \hat{\phi}(\alpha))$$

for $\bar{n} \in \Delta_{i-1}$, $\alpha \in F$ where $\bar{\chi} = p_i \circ \chi : F \rightarrow \Delta_{i-1}$, $\hat{\phi} : F \rightarrow \text{Aut}(N_{i-1})$;

$$\hat{\phi}(\alpha)(\bar{x}) = \overline{\bar{\phi}(\alpha)(x)}.$$

Note that $\bar{\phi}(\alpha)(\mathbf{R}) = \mathbf{R}$. As the action (π_{i-1}, N_{i-1}) is properly discontinuous and π_{i-1} is torsionfree, the representation $\hat{\rho}_i : \pi_{i-1} \rightarrow E(N_{i-1})$ is faithful. (Note that $\hat{\rho}_i|_{\Delta_{i-1}} = \text{id}$.) Thus we obtain an equivariant fibration:

$$(3.9) \quad (\mathbb{Z}, \mathbf{R}) \longrightarrow (\rho_i(\pi_i), \tilde{N}_i) \xrightarrow{\nu_i} (\hat{\rho}_i(\pi_{i-1}), N_{i-1}).$$

Suppose by induction that (π_{i-1}, X_{i-1}) is equivariantly diffeomorphic to the infranil-action $(\hat{\rho}_i(\pi_{i-1}), N_{i-1})$ as above. We have two Seifert fibrations from (1.3) where

$$(\mathbb{Z}, \mathbf{R}) \rightarrow (\pi_i, X_i) \xrightarrow{p_i} (\pi_{i-1}, X_{i-1})$$

and (3.9) where

$$(\mathbb{Z}, \mathbf{R}) \rightarrow (\rho_i(\pi_i), \tilde{N}_i) \xrightarrow{\nu_i} (\hat{\rho}_i(\pi_{i-1}), N_{i-1}).$$

As $\rho_i : \pi_i \rightarrow \rho_i(\pi_i)$ is isomorphic such that $\rho_i|_{\mathbb{Z}} = \text{id}$, the Seifert rigidity implies that (π_i, X_i) is equivariantly diffeomorphic to $(\rho_i(\pi_i), \tilde{N}_i)$. This shows the induction step. Let $M = X/\pi$. Then (π, X) is equivariantly diffeomorphic to an infra-nilaction $(\rho(\pi), \tilde{N})$ for which $\rho : \pi \rightarrow E(\tilde{N})$ is a faithful representation.

We have shown that M is diffeomorphic to an infranilmanifold $\tilde{N}/\rho(\pi)$. According to Cases I, II (stated in Theorem 1.2), we prove that \tilde{N} is isomorphic to a vector space for Case I or \tilde{N} is a nilpotent Lie group but not a vector space for Case II respectively.

Case I. As every cocycle of $H^2_\phi(\pi_{i-1}, \mathbb{Z})$ representing a group extension (3.2) is finite, the cocycle in $H^2(\Delta_{i-1}, \mathbb{Z})$ for the induced extension of (3.3) that $1 \rightarrow \mathbb{Z} \rightarrow \tilde{\Delta}_i \rightarrow \Delta_{i-1} \rightarrow 1$ is also finite. By induction, suppose that Δ_{i-1} is isomorphic to a free abelian group \mathbb{Z}^{i-1} . Then the cocycle in $H^2(\mathbb{Z}^{i-1}, \mathbb{Z})$ is zero, so $\tilde{\Delta}_i$ is isomorphic to a free abelian

group \mathbb{Z}^i . Hence the nilpotent Lie group N_i is isomorphic to the vector space \mathbb{R}^i . This shows the induction step. In particular, π_i is isomorphic to a Bieberbach group $\rho_i(\pi_i) \leq E(\mathbb{R}^i)$. As a consequence X/π is diffeomorphic to a Riemannian flat manifold $\mathbb{R}^n/\rho(\pi)$.

Case II. Suppose that π_{i-1} is virtually free abelian until $i-1$ and the cocycle $[f] \in H_\phi^2(\pi_{i-1}, \mathbb{Z})$ representing a group extension $1 \rightarrow \mathbb{Z} \rightarrow \pi_i \rightarrow \pi_{i-1} \rightarrow 1$ is of infinite order in $H_\phi^2(\pi_{i-1}, \mathbb{Z})$. Note that π_{i-1} contains a torsionfree normal free abelian subgroup \mathbb{Z}^{i-1} . As in (3.3), there is a central group extension of $\tilde{\Delta}_i$:

$$(3.10) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \longrightarrow & \pi_{i-1} \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow i \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_i & \longrightarrow & \mathbb{Z}^{i-1} \longrightarrow 1 \end{array}$$

where $[\pi_{i-1} : \mathbb{Z}^{i-1}] < \infty$. Recall that there is a transfer homomorphism $\tau : H^2(\mathbb{Z}^{i-1}, \mathbb{Z}) \rightarrow H_\phi^2(\pi_{i-1}, \mathbb{Z})$ such that $\tau \circ i^* = [\pi_{i-1} : \mathbb{Z}^{i-1}] : H_\phi^2(\pi_{i-1}, \mathbb{Z}) \rightarrow H_\phi^2(\pi_{i-1}, \mathbb{Z})$, see [2, (9.5) Proposition p.82] for example. The restriction $i^*[f]$ gives the bottom extension sequence of (3.10). If $i^*[f] = 0 \in H^2(\mathbb{Z}^2, \mathbb{Z})$, then $0 = \tau \circ i^*[f] = [\pi_{i-1} : \mathbb{Z}^{i-1}][f] \in H_\phi^2(\pi_{i-1}, \mathbb{Z})$. So $i^*[f] \neq 0$. Therefore $\tilde{\Delta}_i$ (respectively \tilde{N}_i) is not abelian (respectively not isomorphic to a vector space). As a consequence, \tilde{N} is a simply connected (non-abelian) nilpotent Lie group. \square

In order to study S^1 -fibred nilBott manifolds further, we introduce the following definition:

Definition 3.1. If an S^1 -fibred nilBott manifold M satisfies Case I (respectively Case II) of Theorem 1.2, then M is said to be an S^1 -fibred nilBott manifold of finite type (respectively of infinite type). Apparently there is no intersection between finite type and infinite type. And S^1 -fibred nilBott manifolds are of finite type until dimension 2.

Remark 3.2. Let M be an S^1 -fibred nilBott manifold of finite type, then $\rho(\pi)$ is a Bieberbach group (cf. Theorem 1.2). By the Bieberbach Theorem, $\rho(\pi)$ satisfies a group extension

$$(3.11) \quad 1 \rightarrow \mathbb{Z}^n \rightarrow \rho(\pi) \rightarrow H \rightarrow 1$$

where $\mathbb{Z}^n = \rho(\pi) \cap \mathbb{R}^n$, and H is the holonomy group of $\rho(\pi)$. We may identify $\rho(\pi)$ with π whenever π is torsionfree.

Proposition 3.3. Suppose M is an S^1 -fibred nilBott manifold of finite type. Then the holonomy group of π is isomorphic to the power of cyclic group of order two $(\mathbb{Z}_2)^s$ in $(0 \leq s \leq n)$.

Proof. Let M be an S^1 -fibred nilBott manifold of finite type. From (3.2) recall a group extension

$$(3.12) \quad 1 \rightarrow \mathbb{Z} \rightarrow \pi_i \xrightarrow{p_i} \pi_{i-1} \rightarrow 1$$

which associates to the equivariant fibration:

$$(\mathbb{Z}, \mathbb{R}) \rightarrow (\pi_i, \tilde{N}_i) \xrightarrow{p_i} (\pi_{i-1}, N_{i-1}).$$

If f is a cocycle in $H_\phi^2(\pi_{i-1}, \mathbb{Z})$ for Case I representing (3.12), then there exists a map $\lambda : \pi_{i-1} \rightarrow \mathbb{R}$ such that

$$(3.13) \quad f(\alpha, \beta) = \bar{\phi}(\alpha)(\lambda(\beta)) + \lambda(\alpha) - \lambda(\alpha\beta) \quad (\alpha, \beta \in \pi_{i-1})$$

(see [3]). Moreover let $(n, \alpha) \in \pi_i$ and $(x, w) \in \tilde{N}_i = \mathbb{R} \times N_{i-1}$, then the action of π_i is given by

$$(3.14) \quad (n, \alpha)(x, w) = (n + \bar{\lambda}(\alpha)(x) + \lambda(\alpha), \alpha w)$$

(Remark that $n \in \mathbb{Z}$, $\alpha \in \pi_{i-1}$ and see (2.3).) As we have shown in Case I of Theorem 1.2, (π_i, \tilde{N}_i) is a Bieberbach group action. Let $(\pi_{i-1}, N_{i-1}) = (\pi_{i-1}, \mathbb{R}^{i-1})$ where $\pi_{i-1} \leq E(i-1) = \mathbb{R}^{i-1} \rtimes O(i-1)$ is a Bieberbach group such that

$$\alpha w = b_\alpha + A_\alpha w \quad (w \in \mathbb{R}^{m-1})$$

here $b_\alpha \in \mathbb{R}^m$, $A_\alpha \in O(i-1)$ in the above action of (3.14).

Let $L : E(i-1) \rightarrow O(i-1)$ be the linear holonomy homomorphism. Suppose inductively that $L(\pi_{i-1}) = \{A_\alpha \mid \alpha \in \pi_{i-1}\} \leq (\mathbb{Z}_2)^{i-1}$. Here

$$(3.15) \quad (\mathbb{Z}_2)^{i-1} = \left\{ \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} \right\} \leq O(i-1).$$

Then the above action (3.14) has the formula:

$$(3.16) \quad (n, \alpha) \begin{bmatrix} x \\ w \end{bmatrix} = \left(\begin{pmatrix} n + \lambda(\alpha) \\ b_\alpha \end{pmatrix}, \begin{pmatrix} \bar{\phi}(\alpha) & 0 \\ 0 & A_\alpha \end{pmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix},$$

where $\begin{bmatrix} x \\ w \end{bmatrix} \in \tilde{N}_i = \mathbb{R} \times \mathbb{R}^{i-1} = \mathbb{R}^i$. It follows $(n, \alpha) \in E(i)$. Since $\bar{\phi} : \pi_{i-1} \rightarrow \{\pm 1\} \leq \text{Aut}(\mathbb{R})$ is a unique extension of $\phi : \pi_{i-1} \rightarrow \text{Aut}(\mathbb{Z}) = \{\pm 1\}$, we see that the H_i isomorphism $(\mathbb{Z}_2)^s$, $(0 \leq s \leq i)$. This proves the induction step. \square

Corollary 3.4. *Each S^1 -fibred nilBott manifold of finite type M_i admits a homologically injective T^k -action where $k = \text{Rank } H_1(M_i)$. Moreover, the action is maximal, i.e. $k = \text{Rank } C(\pi_i)$.*

Proof. We suppose by induction that there is a *homologically injective* maximal T^{k-1} -action on $M_{i-1} = T^{i-1}/H$ such that $k-1 = \text{Rank } H_1(M_{i-1}) = \text{Rank } C(\pi_{i-1})$ ($k-1 > 0$). Since π_i, π_{i-1} are Bieberbach groups, there are two group extensions

$$1 \rightarrow \mathbb{Z}^i \rightarrow \pi_i \xrightarrow{h_i} H_i \rightarrow 1$$

$$1 \rightarrow \mathbb{Z}^{i-1} \rightarrow \pi_{i-1} \xrightarrow{h_{i-1}} H_{i-1} \rightarrow 1$$

where H_i, H_{i-1} are holonomy groups of π_i, π_{i-1} , respectively and $\mathbb{Z}^i = \pi_i \cap \mathbb{R}^i, \mathbb{Z}^{i-1} = \pi_{i-1} \cap \mathbb{R}^{i-1}$. We have a following diagram

$$(3.17) \quad \begin{array}{ccccccc} & & & 1 & & 1 & \\ & & & \downarrow & & \downarrow & \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^i & \longrightarrow & \mathbb{Z}^{i-1} \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_i & \xrightarrow{p_i} & \pi_{i-1} \longrightarrow 1 \\ & & & & \downarrow h_i & & \downarrow h_{i-1} \\ & & & & H_i & \xlongequal{\quad} & H_{i-1} \\ & & & & \downarrow & & \downarrow \\ & & & & 1 & & 1 \end{array}$$

Let $p : \mathbb{R}^i = \mathbb{R} \times \mathbb{R}^{i-1} \rightarrow T^i = S^1 \times T^{i-1}$ be the canonical projection such that $\text{Ker } p = \mathbb{Z}^i = \pi_i \cap \mathbb{R}^i$. By Proposition 3.3, $H_i = (\mathbb{Z}_2)^s$ for some s ($1 \leq s \leq i$). The action (π_i, \mathbb{R}^i) induces an isometric action (H_i, T^i) from (3.16). We may represent the action as the following

$$(3.18) \quad \hat{\alpha} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_i \end{pmatrix} = \begin{pmatrix} t_{\hat{\alpha}} \cdot \psi(\hat{\alpha})(z_1) \\ z_2' \\ \vdots \\ z_i' \end{pmatrix}$$

here $\hat{\alpha} = h_i((n, \alpha)) \in H_i$, $t_{\hat{\alpha}} = p(n + \lambda(\alpha)) \in S^1$, and $\psi : H_i \rightarrow \{\pm 1\}$,

$$(3.19) \quad \psi(\hat{\alpha})(z_1) = \begin{cases} z_1 & \text{if } \bar{\phi}(\alpha) = 1 \\ \bar{z}_1 & \text{if } \bar{\phi}(\alpha) = -1. \end{cases}$$

Note that $(t_{\hat{\alpha}})^2 = p(n + \lambda(\alpha))p(n + \lambda(\alpha)) = p(2n + 2\lambda(\alpha))$. Suppose $\bar{\phi}(\alpha) = 1$. By (3.16),

$$(3.20) \quad (n, \alpha)^2 \begin{bmatrix} x \\ w \end{bmatrix} = \left(\begin{pmatrix} 2n + 2\lambda(\alpha) \\ b_{\alpha} + A_{\alpha}w \end{pmatrix}, \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix}.$$

Since $2n + 2\lambda(\alpha) \in \mathbb{Z}$, then $(t_{\hat{\alpha}})^2 = 1$ i.e. $t_{\hat{\alpha}} = \pm 1$.

If $H_i = \text{Ker } \psi$, it follows from (3.21) that the S^1 -action on $T^i = S^1 \times T^{i-1}$ as left translations induces an S^1 -action on $M_i = T^i/H_i$ so that T^k -action on $M_i = T^i/H_i$ follows

$$(3.21) \quad \begin{pmatrix} t \\ t' \end{pmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_i \end{bmatrix} = \begin{bmatrix} t \cdot z_1 \\ t' \cdot \begin{pmatrix} z_2 \\ \vdots \\ z_i \end{pmatrix} \end{bmatrix}$$

where $(t, t') \in S^1 \times T^{k-1}$, $[z_1, \dots, z_i] \in M_i = T^i/H_i$. On the other hand, if there is an element $\hat{\alpha}$ of H_i which $\psi(\hat{\alpha})(z) = \bar{z}$, then M_i admits a T^{k-1} -action by the induction hypothesis. The group extension (3.12) gives rise to a group extension:

$$(3.22) \quad 1 \rightarrow \mathbb{Z}/[\pi_i, \pi_i] \cap \mathbb{Z} \rightarrow \pi_i/[\pi_i, \pi_i] \xrightarrow{\nu_i} \pi_{i-1}/[\pi_{i-1}, \pi_{i-1}] \rightarrow 1.$$

As in the proof of Proposition 3.3, $[(0, \alpha), (n, 1)] = ((\phi(\alpha) - 1)(n), 1)$. It follows that $[\pi_i, \pi_i] \cap \mathbb{Z} = \{1\}$ or $[\pi_i, \pi_i] \cap \mathbb{Z} = 2\mathbb{Z}$ according to whether $H_i = \text{Ker } \psi$ or not. So (3.22) becomes

$$(3.23) \quad 1 \longrightarrow \mathbb{Z} \longrightarrow H_1(M_i) \xrightarrow{\nu_i} H_1(M_{i-1}) \longrightarrow 1,$$

or

$$(3.24) \quad 1 \longrightarrow \mathbb{Z}_2 \longrightarrow H_1(M_i) \xrightarrow{\nu_i} H_1(M_{i-1}) \longrightarrow 1.$$

For (3.23), it follows $k = \text{Rank } H_1(M_i)$ for which M_i admits a homologically injective T^k -action as above. For (3.24), $k - 1 = \text{Rank } H_1(M_i)$ and M_i admits a homologically injective T^{k-1} -action by the induction hypothesis.

Suppose $H_i = \text{Ker } \psi$. Noting that the group extension $1 \rightarrow \mathbb{Z} \rightarrow \pi_i \xrightarrow{p_i} \pi_{i-1} \rightarrow 1$ is a central extension, we obtain a group extension:

$$1 \rightarrow \mathbb{Z} \rightarrow C(\pi_i) \xrightarrow{p_i} p_i(C(\pi_i)) \rightarrow 1.$$

On the other hand, since M_i admits the above T^k -action, $\mathbb{Z}^k \subset C(\pi_i)$. Let $\text{Rank } C(\pi_i) = k + l$, ($l = 0, 1, 2, \dots$), then $\mathbb{Z}^{k+l-1} \subset p_i(C(\pi_i))$. By the induction hypothesis, $k - 1 = \text{Rank } C(\pi_{i-1}) \geq \text{Rank } p_i(C(\pi_i))$. Therefore as $l = 0$, $k = \text{Rank } C(\pi_i)$.

Assume that there exists an element $\hat{\alpha} \in H_i$ such that $\psi(\hat{\alpha})(z) = \bar{z}$.

It is easy to check that $\mathbb{Z} \cap C(\pi_i) = \phi$, i.e. $C(\pi_i) \leq C(\pi_{i-1})$ and since M_i admits T^{k-1} -action, $\mathbb{Z}^{k-1} \leq C(\pi_i)$. By the induction hypothesis, $k-1 = \text{Rank } C(\pi_i)$.

Therefore in each case the torus action is maximal. \square

4. 3-DIMENSIONAL S^1 -FIBRED NILBOTT TOWERS

By the definition of S^1 -fibred nilBott manifold M_n , M_2 is either a torus T^2 or a Klein bottle K so that M_2 is a Riemannian flat manifold.

4.1. 3-dimensional S^1 -fibred nilBott manifolds of finite type.

Any 3-dimensional S^1 -fibred nilBott manifold M_3 of finite type is a Riemannian flat manifold. It is known that there are just 10-isomorphism classes $\mathcal{G}_1, \dots, \mathcal{G}_6, \mathcal{B}_1, \dots, \mathcal{B}_4$ of 3-dimensional Riemannian flat manifolds. (Refer to the classification of 3-dimensional Riemannian flat manifolds by Wolf [16].) In particular, for Riemannian flat 3-manifolds corresponding to \mathcal{B}_2 and \mathcal{B}_4 , we have shown that there are two S^1 -fibred nilBott towers: $\mathcal{B}_2 \rightarrow K \rightarrow S^1 \rightarrow \{\text{pt}\}$ and $\mathcal{B}_4 \rightarrow K \rightarrow S^1 \rightarrow \{\text{pt}\}$ in [13]. Remark that every real Bott manifold is an S^1 -fibred nilBott manifold of finite type and \mathcal{B}_2 and \mathcal{B}_4 are not real Bott manifolds. And the following Proposition 4.1 have been proved. See [13] for details.

Proposition 4.1. *The 3-dimensional S^1 -fibred nilBott manifold of finite type are those of $\mathcal{G}_1, \mathcal{G}_2, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4$.*

4.2. 3-dimensional S^1 -fibred nilBott manifolds of infinite type.

Any 3-dimensional S^1 -fibred nilBott manifold M_3 of infinite type is an infranil-Heisenberg manifold. The 3-dimensional simply connected nilpotent Lie group N_3 is isomorphic to the Heisenberg Lie group \mathbf{N} which is the product $\mathbb{R} \times \mathbb{C}$ with group law:

$$(x, z) \cdot (y, w) = (x + y - \text{Im} \bar{z}w, z + w).$$

Then the maximal compact Lie subgroup of $\text{Aut}(\mathbf{N})$ is $U(1) \rtimes \langle \tau \rangle$ which acts on \mathbf{N}

$$(4.1) \quad \begin{aligned} e^{i\theta}(x, z) &= (x, e^{i\theta}z), \quad (e^{i\theta} \in U(1)). \\ \tau(x, z) &= (-x, \bar{z}). \end{aligned}$$

A 3-dimensional compact infranilmanifold is obtained as a quotient \mathbf{N}/Γ where Γ is a torsionfree discrete uniform subgroup of $E(\mathbf{N}) = \mathbf{N} \rtimes (U(1) \rtimes \langle \tau \rangle)$. (See [4].)

Let

$$S^1 \rightarrow M_3 \rightarrow M_2$$

be an S^1 -fibred nilBott manifold of infinite type which has a group extension $1 \rightarrow \mathbb{Z} \rightarrow \pi_3 \rightarrow \pi_2 \rightarrow 1$. Since $\mathbf{R} \subset \mathbf{N}$ is the center, there is a commutative diagram of central extensions (cf. (3.17)):

$$(4.2) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \tilde{\Delta}_3 & \longrightarrow & \Delta_2 \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbf{R} & \longrightarrow & \mathbf{N} & \longrightarrow & \mathbb{C} \longrightarrow 1. \end{array}$$

Using this, we obtain an embedding:

$$(4.3) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \longrightarrow & \pi_2 \longrightarrow 1 \\ & & \downarrow & & \rho \downarrow & & \hat{\rho} \downarrow \\ 1 & \longrightarrow & \mathbf{R} & \longrightarrow & \mathbf{E}(\mathbf{N}) & \longrightarrow & \mathbb{C} \rtimes (\mathrm{U}(1) \rtimes \langle \tau \rangle) \longrightarrow 1. \end{array}$$

Note that $\mathbb{C} \rtimes (\mathrm{U}(1) \rtimes \langle \tau \rangle) = \mathbb{R}^2 \rtimes \mathrm{O}(2) = \mathrm{E}(2)$. Since $\mathbb{R} \cap \pi_3 = \mathbb{Z}$ from (4.3), $\hat{\rho}(\pi_2)$ is a Bieberbach group in $\mathrm{E}(2)$ so that $\mathbb{R}^2/\hat{\rho}(\pi_2)$ is either T^2 or K .

Case (i). Suppose that the holonomy group of π_3 is trivial. Since $L(\pi_3) = \{1\}$ in $\mathrm{U}(1) \rtimes \langle \tau \rangle$, it is noted that $\pi_3 = \tilde{\Delta}_3$ from (4.3) and (4.2). As $\tilde{\Delta}_3 \leq \mathbf{N}$, $\tilde{\Delta}_3$ is isomorphic to $\Delta(k)$ defined below.

Let $k \in \mathbb{Z}$ and define $\Delta(k)$ to be a subgroup of \mathbf{N} generated by

$$c = (2k, 0), a = (0, k), b = (0, k\mathbf{i}).$$

Put $Z = \langle c \rangle$ which is a central subgroup of $\Delta(k)$. It is easy to see that

$$(4.4) \quad [a, b] = c^{-k}.$$

Since \mathbf{R} is the center of \mathbf{N} , we have a principal bundle

$$S^1 = \mathbf{R}/Z \rightarrow \mathbf{N}/\Delta(k) \rightarrow \mathbb{C}/\mathbb{Z}^2.$$

Then the euler number of the fibration is $\pm k$. (See [12] for example.)

Case (ii). Suppose that the holonomy group of π_3 is nontrivial. Then we note that $L(\pi_3) = \mathbb{Z}_2 \leq \mathrm{U}(1) \rtimes \langle \tau \rangle$, but not in $\mathrm{U}(1)$. By (3.17) $L(\pi_3) = L(\pi_2)$, so first we note that $L(\pi_2)$ is not contained in $\mathrm{U}(1)$. Suppose that (b, A) is a element of $\pi_2 \leq \mathbb{R}^2 \rtimes \mathrm{O}(2)$. Then for any $x \in \mathbb{R}^2$, $(b, A)x \neq x$, because the action of π_2 on \mathbb{R}^2 is free. Therefore determinant of $(A - E)$ is zero. This implies that if $A \in \mathrm{SO}(2)$, then $A = E$. So $L(\pi_2) = L(\pi_3)$ not in $\mathrm{U}(1)$. Suppose that there exists an element $g \in \pi_3$ such that $L(g) = (e^{i\theta}, \tau) \in \mathrm{U}(1) \rtimes \langle \tau \rangle$. Noting

(4.1), it follows $L(g)^2 = 1$. Then $L(\pi_3) = (U(1) \cap L(\pi_3)) \cdot \langle L(g) \rangle$. Let $\pi'_3 = L^{-1}(U(1) \cap L(\pi_3)) \leq \pi_3$ which has the commutative diagram:

$$(4.5) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \xrightarrow{p_3} & \pi_2 \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi'_3 & \longrightarrow & \pi'_2 \longrightarrow 1. \end{array}$$

Since π'_2 also acts on \mathbb{R}^2 free, $L(\pi'_3) = U(1) \cap L(\pi_3) = \{1\}$. Hence $L(\pi_3) = \mathbb{Z}_2 = \langle L(g) \rangle$. Note that M_2 is the Klein bottle K , since $L(\pi_2) = \mathbb{Z}_2$. Let $n = (x, 0)$ be a generator of $\mathbb{Z} \leq \mathbf{N}$. Choose $h \in \pi_3$ with $L(h) = 1$ such that the subgroup $\langle p_3(g), p_3(h) \rangle$ is the fundamental group of K . It has a relation $p_3(g)p_3(h)p_3(g)^{-1} = p_3(h)^{-1}$. Then $\langle n, g, h \rangle$ is isomorphic to π_3 . In particular, those generators satisfy

$$(4.6) \quad \begin{aligned} ghg^{-1} &= n^k h^{-1} \ (\exists k \in \mathbb{Z}), \\ gng^{-1} &= L(g)n = \tau n = n^{-1}, \quad hnh^{-1} = L(h)n = n. \end{aligned}$$

On the other hand, let $\Gamma(k)$ be a subgroup of $E(\mathbf{N})$ generated by

$$(4.7) \quad n = ((k, 0), I), \alpha = \left((0, \frac{k}{2}), \tau \right), \beta = ((0, k\mathbf{i}), I).$$

Note that $\alpha^2 = ((0, k), I)$. Then it is easily checked that

$$(4.8) \quad \alpha n \alpha^{-1} = n^{-1}, \alpha \beta \alpha^{-1} = n^k \beta^{-1}, \beta n \beta^{-1} = n.$$

$$(4.9) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{R} & \longrightarrow & E(\mathbf{N}) & \longrightarrow & \mathbb{C} \rtimes (U(1) \rtimes \langle \tau \rangle) \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \langle n \rangle & \longrightarrow & \Gamma(k) & \longrightarrow & \langle \hat{\alpha}, \hat{\beta} \rangle \longrightarrow 1. \end{array}$$

Then the subgroup generated by $\hat{\alpha}^2, \hat{\beta}$ is isomorphic to the subgroup of translations of \mathbb{R}^2 ; $t_1 = \begin{pmatrix} k \\ 0 \end{pmatrix}, t_2 = \begin{pmatrix} 0 \\ k \end{pmatrix}$. Let $T^2 = \mathbb{R}^2 / \langle t_1, t_2 \rangle$. Then it is easy to see that the element $\gamma = [\hat{\alpha}]$ of order 2 acts on T^2 as

$$(4.10) \quad \gamma(z_1, z_2) = (-z_1, \bar{z}_2).$$

As a consequence, $\mathbb{R}^2 / \langle \hat{\alpha}, \hat{\beta} \rangle = T^2 / \langle \gamma \rangle$ turns out to be K . So $M_3 = \mathbf{N} / \Gamma(k)$ is an S^1 -fibred nilBott manifold:

$$S^1 \rightarrow \mathbf{N} / \Gamma(k) \rightarrow K$$

where $S^1 = \mathbf{R} / \langle n \rangle$ is the fiber (but not an action).

Compared (4.6) with $\Gamma(k)$, π_3 is isomorphic to $\Gamma(k)$ with the following commutative arrows of isomorphisms:

$$(4.11) \quad \begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi_3 & \longrightarrow & \pi_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \langle n \rangle & \longrightarrow & \Gamma(k) & \longrightarrow & \langle \hat{\alpha}, \hat{\beta} \rangle & \longrightarrow & 1. \end{array}$$

As both (π_3, X_3) and $(\Gamma(k), \mathbf{N})$ are Seifert actions, the isomorphism of (4.11) implies that they are equivariantly diffeomorphic, i.e. $M_3 = X_3/\pi_3 \cong \mathbf{N}/\Gamma(k)$.

This shows the following.

Proposition 4.2. *A 3-dimensional an S^1 -fibred nilBott manifold M_3 of infinite type is either a Heisenberg nilmanifold $\mathbf{N}/\Delta(k)$ or an infranilmanifold $\mathbf{N}/\Gamma(k)$.*

5. REALIZATION

Let $Q = \pi_1(K)$ be the fundamental group of K . Q has a presentation:

$$(5.1) \quad \{g, h \mid ghg^{-1} = h^{-1}\}.$$

A group extension $1 \rightarrow \mathbb{Z} \rightarrow \pi \rightarrow Q \rightarrow 1$ for any 3-dimensional S^1 -fibred nilBott manifold over K represents a 2-cocycle in $H_\phi^2(Q, \mathbb{Z})$ for some representation ϕ . Conversely, given a representation ϕ , we may show any element of $H_\phi^2(Q, \mathbb{Z})$ can be realized as an S^1 -fibred nilBott manifold.

We must consider following cases of a representation ϕ :

Case 1. $\phi(g) = 1, \phi(h) = 1,$

Case 2. $\phi(g) = 1, \phi(h) = -1,$

Case 3. $\phi(g) = -1, \phi(h) = 1,$

Case 4. $\phi(g) = -1, \phi(h) = -1.$

Let ϕ_i ($i = 1, 2, 3, 4$) be the representation ϕ of the **Case** i above. A 2-cocycle $[f_k] \in H_{\phi_i}^2(Q, \mathbb{Z})$ gives rise to a group extension

$$1 \rightarrow \mathbb{Z} \rightarrow {}_i\pi(k) \xrightarrow{p} G \rightarrow 1,$$

where ${}_i\pi(k)$ is generated by \tilde{g}, \tilde{h}, n such that $\langle n \rangle = \mathbb{Z}$, $p(\tilde{g}) = g$, $p(\tilde{h}) = h$. By (5.1),

$$(5.2) \quad \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}.$$

for some $k \in \mathbb{Z}$. Note that $[f_0] = 0$.

Case 1: Since ϕ_1 is trivial, $H_{\phi_1}^2(Q, \mathbb{Z}) = H^2(Q, \mathbb{Z}) \approx H^2(K, \mathbb{Z}) \approx \mathbb{Z}_2$. Moreover ${}_1\pi(k)$ satisfies the presentation:

$$(5.3) \quad \tilde{g}n\tilde{g}^{-1} = n, \tilde{h}n\tilde{h}^{-1} = n, \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}.$$

Lemma 5.1. *The groups ${}_1\pi(0)$, ${}_1\pi(1)$ are isomorphic to $\pi_1(\mathcal{B}_1)$, $\pi_1(\mathcal{B}_2)$ respectively.*

Proof. First we discuss about ${}_1\pi(0)$. Let $\tilde{g}, \tilde{h}, n \in {}_1\pi(0)$ be as above. Put $\varepsilon = \tilde{g}$, $t_1 = \tilde{g}^2$, $t_2 = n$, and $t_3 = \tilde{h}$. Note that the group which generated by $\varepsilon, t_1, t_2, t_3$ coincides with ${}_1\pi(0)$. Using the relation (5.3),

$$\begin{aligned} \varepsilon^2 &= t_1, \\ \varepsilon t_2 \varepsilon^{-1} &= \tilde{g}\tilde{h}\tilde{g}^{-1} = \tilde{h}^{-1} = t_2^{-1}, \\ \varepsilon t_3 \varepsilon^{-1} &= \tilde{g}n\tilde{g}^{-1} = n = t_3. \end{aligned}$$

Compared this relation with $\pi_1(\mathcal{B}_1)$, ${}_1\pi(0)$ is isomorphic to $\pi_1(\mathcal{B}_1)$. (in the Wolf's notation [16])

Second, about ${}_1\pi(1)$. Let $\tilde{g}, \tilde{h}, n \in {}_1\pi(1)$ be as above. Put $\varepsilon = \tilde{g}$, $t_1 = \tilde{g}^2$, $t_2 = \tilde{g}^{-2}n$, and $t_3 = h$. The group which generated by $\varepsilon, t_1, t_2, t_3$ coincides with ${}_1\pi(1)$. By using the relation (5.3),

$$\begin{aligned} \varepsilon^2 &= t_1, \\ \varepsilon t_2 \varepsilon^{-1} &= \tilde{g}\tilde{g}^{-2}n\tilde{g}^{-1} = \tilde{g}^{-1}n\tilde{g}^{-1} = \tilde{g}^{-2}n = t_1, \\ \varepsilon t_3 \varepsilon^{-1} &= \tilde{g}h\tilde{g}^{-1} = \tilde{g}^2\tilde{g}^{-2}n\tilde{h}^{-1} = t_1 t_2 t_3^{-1}. \end{aligned}$$

This implies that ${}_1\pi(1)$ is isomorphic to $\pi_1(\mathcal{B}_2)$. (See [16]) □

Remark that the fundamental group $\pi_1(\mathcal{B}_2)$ is isomorphic to ${}_1\pi(1)$ so we have a group extension which represents the $[f_1] \in H_{\phi_1}^2(Q, \mathbb{Z})$ with 2-torsion. Therefore, if $[f_k] \neq 0$, then $k = 1$.

Case 2: Let $\phi_2(g) = 1, \phi_2(h) = -1$, then ${}_2\pi(k)$ has the following presentation.

$$(5.4) \quad \tilde{g}n\tilde{g}^{-1} = n, \tilde{h}n\tilde{h}^{-1} = n^{-1}, \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}.$$

for some $k \in \mathbb{Z}$,

Proposition 5.2. *The groups ${}_2\pi(0)$, ${}_2\pi(1)$ are isomorphic to $\pi_1(\mathcal{B}_3)$, $\pi_1(\mathcal{B}_4)$ respectively.*

Proof. Let $\tilde{g}, \tilde{h}, n \in {}_2\pi(0)$ be as before. put $\alpha = \tilde{h}\tilde{g}$, $\varepsilon = \tilde{h}^{-1}$, $t_1 = \tilde{g}^2$, $t_2 = \tilde{h}^{-2}$, and $t_3 = n$. Note that the group generated by $\alpha, \varepsilon, t_1, t_2, t_3$ coincides with ${}_2\pi(0)$. Using the relation (5.4),

$$\begin{aligned}\tilde{\alpha}^2 &= (\tilde{h}\tilde{g})^2 = \tilde{h}\tilde{h}^{-1}\tilde{g}\tilde{g} = \tilde{g}^2 = t_1, \\ \varepsilon^2 &= t_2, \\ \varepsilon\alpha\varepsilon^{-1} &= \tilde{h}^{-1}\tilde{h}\tilde{g}\tilde{h} = \tilde{h}^{-1}\tilde{g} = t_2\alpha, \\ \alpha t_2\alpha^{-1} &= \tilde{h}\tilde{g}\tilde{h}^{-2}\tilde{g}^{-1}\tilde{h}^{-1} = \tilde{h}^{-2} = t_2^{-1}, \\ \alpha t_3\alpha^{-1} &= \tilde{h}\tilde{g}n\tilde{g}^{-1}\tilde{h}^{-1} = n^{-1} = t_3^{-1}, \\ \varepsilon t_1\varepsilon^{-1} &= \tilde{h}^{-1}\tilde{g}^2\tilde{h} = \tilde{h}^{-1}\tilde{g}\tilde{h}^{-1}\tilde{g} = \tilde{h}^{-1}\tilde{h}\tilde{g}\tilde{g} = \tilde{g}^2 = t_1, \\ \varepsilon t_3\varepsilon^{-1} &= \tilde{h}^{-1}n\tilde{h} = n^{-1} = t_3^{-1}.\end{aligned}$$

This relation correspond to of $\pi_1(\mathcal{B}_3)$. (See [16]). So ${}_2\pi(0)$ is isomorphic to $\pi_1(\mathcal{B}_3)$.

Let $\tilde{g}, \tilde{h}, n \in {}_2\pi(1)$ be as above. Put $\alpha = \tilde{h}\tilde{g}$, $\varepsilon = n^{-1}\tilde{h}^{-1}$, $t_1 = n^{-1}\tilde{g}^2$, $t_2 = \tilde{h}^{-2}$, and $t_3 = n^{-1}$. Using the relation (5.4), we obtain a presentation:

$$\begin{aligned}\tilde{\alpha}^2 &= (\tilde{h}\tilde{g})^2 = \tilde{h}n\tilde{h}^{-1}\tilde{g}\tilde{g} = n^{-1}\tilde{g}^2 = t_1, \\ \varepsilon^2 &= t_2, \\ \varepsilon\alpha\varepsilon^{-1} &= n^{-1}\tilde{h}^{-1}\tilde{h}\tilde{g}\tilde{h}n = \tilde{h}^{-1}\tilde{g}n = t_2t_3\alpha, \\ \alpha t_2\alpha^{-1} &= \tilde{h}\tilde{g}\tilde{h}^{-2}\tilde{g}^{-1}\tilde{h}^{-1} = \tilde{h}^{-2} = t_2^{-1}, \\ \alpha t_3\alpha^{-1} &= \tilde{h}\tilde{g}n^{-1}\tilde{g}^{-1}\tilde{h}^{-1} = n = t_3^{-1}, \\ \varepsilon t_1\varepsilon^{-1} &= n^{-1}\tilde{h}^{-1}n^{-1}\tilde{g}^2\tilde{h}n = n^{-1}\tilde{t}_2^2 = t_1, \\ \varepsilon t_3\varepsilon^{-1} &= n^{-1}\tilde{h}^{-1}n^{-1}\tilde{h}n = n = t_3^{-1}.\end{aligned}$$

This implies that ${}_2\pi(1)$ is isomorphic to \mathcal{B}_4 . (See [16]) □

Proposition 5.3. *Any element of $H_{\phi_2}^2(Q, \mathbb{Z})$ is isomorphic to \mathbb{Z}_2*

Proof. Let Q' be the subgroup of Q which is generated by $g, h^2 \in Q$ with $gh^2g^{-1} = (ghg^{-1})^2 = h^{-2}$. We have a commutative diagram:

$$(5.5) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & {}_2\pi(k) & \xrightarrow{p} & Q \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \pi' & \xrightarrow{p} & Q' \longrightarrow 1 \end{array}$$

where π' is the subgroup of ${}_2\pi(k)$ which is generated by $n, \tilde{g}, \tilde{h}^2$. Note that

$$\tilde{g}\tilde{h}^2\tilde{g}^{-1} = n^k\tilde{h}^{-1}n^k\tilde{h}^{-1} = \tilde{h}^{-2}.$$

Since the subgroup $\langle \tilde{g}, \tilde{h}^2 \rangle$ of π' maps isomorphically onto Q' and a restriction $\phi|_{Q'} = \text{id}$, then $\pi' = \mathbb{Z} \times Q'$. This shows that the restriction homomorphism $\iota^* : H_{\phi_2}^2(Q, \mathbb{Z}) \rightarrow H^2(Q', \mathbb{Z})$ is the zero map, equivalently $\iota^*[f_k] = 0$. Using the transfer homomorphism $\tau : H^2(Q', \mathbb{Z}) \rightarrow H_{\phi_2}^2(Q, \mathbb{Z})$ and by the property $\tau \circ \iota^*([f]) = [Q : Q'] [f] = 2[f]$ ($\forall [f] \in H_{\phi_2}^2(Q, \mathbb{Z})$), we obtain $2[f] = 0$.

On the other hand, from (5.4)

$$(5.6) \quad \begin{aligned} n^k &= \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h} = (0, g)(0, h)(-f_k(g^{-1}, g), g^{-1})(0, h) \\ &= f_k(g, h) + f_k(g^{-1}, g) + f_k(gh, g) + f_k(h^{-1}, h). \end{aligned}$$

Since there exists a 2-cocycle $[f_1]$ from Proposition 5.2,

$$n = f_1(g, h) + f_1(g^{-1}, g) + f_1(gh, g) + f_1(h^{-1}, h),$$

and

$$(5.7) \quad n^k = kf_1(g, h) + kf_1(g^{-1}, g) + kf_1(gh, g) + kf_1(h^{-1}, h).$$

Noting that $[f_k]$ represents of ${}_2\pi(k)$ if and only if f_k satisfies (5.6), the relation (5.7) shows that

$$(5.8) \quad [f_k] = k \cdot [f_1].$$

As the consequence, $H_{\phi_2}^2(Q, \mathbb{Z})$ is isomorphic to \mathbb{Z}_2 . \square

This gives the following result:

Corollary 5.4. *The group extension ${}_2\pi(k)$ is isomorphic to $\pi_1(\mathcal{B}_3)$ or $\pi_1(\mathcal{B}_4)$ accordance with k is odd or even.*

Case 3: The group ${}_3\pi(k)$ has the following presentation. For some $k \in \mathbb{Z}$,

$$(5.9) \quad \tilde{g}n\tilde{g}^{-1} = n^{-1}, \quad \tilde{h}n\tilde{h}^{-1} = n, \quad \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}.$$

Lemma 5.5. *The groups ${}_3\pi(0)$, ${}_3\pi(k)$ are isomorphic to $\pi_1(\mathcal{G}_2)$, $\Gamma(k)$ respectively. (cf. (4.7))*

Proof. Let $\tilde{g}, \tilde{h}, n \in {}_3\pi(0)$ be as before. Put $\alpha = \tilde{g}$, $t_1 = \tilde{g}^2$, $t_2 = \tilde{h}$, and $t_3 = n$. Note that the group generated by α, t_1, t_2, t_3 coincides with ${}_3\pi(0)$. By using the relation (5.9), it is easy to check that:

$$\begin{aligned} \alpha^2 &= t_1, \\ \alpha t_2 \alpha^{-1} &= t_2^{-1}, \\ \alpha t_3 \alpha^{-1} &= t_3^{-1}. \end{aligned}$$

So ${}_3\pi(0)$ is isomorphic to \mathcal{G}_2 . (See [16].)

Suppose $\tilde{g}, \tilde{h}, n \in {}_3\pi(k)$. Put $\alpha = \tilde{g}$, $\beta = \tilde{h}$. This implies that ${}_3\pi(k)$ is isomorphic to $\Gamma(k)$. \square

Proposition 5.6. $H_{\phi_3}^2(G, \mathbb{Z})$ is isomorphic to \mathbb{Z} .

Proof. From Theorem 1.2 and Lemma 5.8, $H_{\phi_3}^2(G, \mathbb{Z})$ is torsionfree. Moreover, it satisfies (5.8). Therefore $H_{\phi_3}^2(G, \mathbb{Z})$ is isomorphic to \mathbb{Z} . \square

Case 4. The group ${}_4\pi(k)$ has the following presentation.

$$(5.10) \quad \tilde{g}n\tilde{g}^{-1} = n^{-1}, \tilde{h}n\tilde{h}^{-1} = n^{-1}, \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}^{-1}.$$

Put $\alpha = gh$. It is easy to check that

$$(5.11) \quad \alpha n \alpha^{-1} = n, \tilde{h} n \tilde{h}^{-1} = n^{-1}, \alpha \tilde{h} \alpha = n^k \tilde{h}^{-1}$$

Noting that ${}_4\pi(k)$ coincide with the group which is generated by α, \tilde{h} and n , we can show that ${}_4\pi(k)$ is isomorphic to ${}_2\pi(k)$.

We have shown that any element of $H_{\phi}^2(Q, \mathbb{Z})$ can be realized as an S^1 -fibred nilBott manifold, and obtain the following table.

		Case 1	Case2 and 4	Case3
	$H_{\phi}^2(Q, \mathbb{Z})$	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
$\pi_1(M_3)$	$[f] = 0$	$\pi_1(\mathcal{B}_1)$	$\pi_1(\mathcal{B}_3)$	$\pi_1(\mathcal{G}_2)$
	$[f] \neq 0$:torsion	$\pi_1(\mathcal{B}_2)$	$\pi_1(\mathcal{B}_4)$	-
	$[f]$:torsionfree	-	-	$\Gamma(k)$

Let $\mathbb{Z}^2 = \pi_1(T^2)$ be the fundamental group of T^2 which is generated by α, β .

Given a representation ϕ , we may show any element of $H_{\phi}^2(\mathbb{Z}^2, \mathbb{Z})$ can be realized as an S^1 -fibred nilBott manifold.

We must consider following cases of a representation ϕ :

Case 5. $\phi(\alpha) = 1$, $\phi(\beta) = 1$,

Case 6. $\phi(\alpha) = 1$, $\phi(\beta) = -1$,

Case 7. $\phi(\alpha) = -1$, $\phi(\beta) = -1$.

Let denote ϕ_i as before. In each case, a 2-cocycle $[f_k] \in H_{\phi_i}^2(\mathbb{Z}^2, \mathbb{Z})$ gives rise to a group extension

$$1 \rightarrow \mathbb{Z} \rightarrow {}_i\pi(k) \xrightarrow{p} \mathbb{Z}^2 \rightarrow 1,$$

where ${}_i\pi(k)$ is generated by $\tilde{\alpha}, \tilde{\beta}, m$ such that $\langle m \rangle = \mathbb{Z}$, $p(\tilde{\alpha}) = \alpha$, $p(\tilde{\beta}) = \beta$. By (5.1),

$$(5.12) \quad \tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1} = m^k\tilde{\beta}.$$

for some $k \in \mathbb{Z}$.

Case5: The group ${}_5\pi(k)$ has the following presentation.

$$(5.13) \quad \tilde{\alpha}m\tilde{\alpha}^{-1} = m, \quad \tilde{\beta}m\tilde{\beta}^{-1} = m, \quad \tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1} = m^k\tilde{\beta}.$$

for some $k \in \mathbb{Z}$, this relation gives the following proposotion. (See (4.4)).

Proposition 5.7. *The groups ${}_5\pi(0)$, ${}_5\pi(k)$ are isomorphic to $\pi_1(T^3)$, $\pi_1(\Delta(-k))$ respectively.*

Case 6: The group ${}_6\pi(k)$ has the following presentation.

$$(5.14) \quad \tilde{\alpha}m\tilde{\alpha}^{-1} = m, \quad \tilde{\beta}m\tilde{\beta}^{-1} = m^{-1}, \quad \tilde{\alpha}\tilde{\beta}\tilde{\alpha}^{-1} = m^k\tilde{\beta}.$$

for some $k \in \mathbb{Z}$.

Proposition 5.8. *The groups ${}_6\pi(0)$, ${}_6\pi(1)$ are isomorphic to $\pi_1(\mathcal{B}_1)$, $\pi_1(\mathcal{B}_2)$ respectively.*

Proof. First let $k = 0$. Put $m = \tilde{h}$, $\tilde{\alpha} = n$, $\tilde{\beta} = \tilde{g}$, then we can check easilly that ${}_6\pi(0)$ is isomorphic to ${}_1\pi(0)$, i.e. $\tilde{g}n\tilde{g}^{-1} = n$, $\tilde{h}n\tilde{h}^{-1} = n$, $\tilde{g}\tilde{h}\tilde{g}^{-1} = \tilde{h}^{-1}$. So ${}_6\pi(0)$ is isomorphic to $\pi_1(\mathcal{B}_1)$.

Second suppose $k = 1$. Put $m = n$, $\tilde{\alpha} = \tilde{g}$, $m^{-1}\tilde{\beta} = \tilde{h}$, then we can check easilly that ${}_6\pi(1)$ is isomorphic to $\pi_1(\mathcal{B}_2)$ in the same way above. \square

Moreover we can obtain the following after the fashion of proof for the Proposition 5.3

Case 7: The group ${}_7\pi(k)$ has the following presentation.

$$(5.15) \quad \tilde{g}n\tilde{g}^{-1} = n^{-1}, \quad \tilde{h}n\tilde{h}^{-1} = n^{-1}, \quad \tilde{g}\tilde{h}\tilde{g}^{-1} = n^k\tilde{h}$$

for some $k \in \mathbb{Z}$. Then it easy check that ${}_7\pi(k)$ is isomorphic to ${}_6\pi(k)$ in the same way for Case4 above.

As a consequence, we obtain a table:

		Case 1	Case2 and 3
	$H_\phi^2(\mathbb{Z}^2, \mathbb{Z})$	\mathbb{Z}	\mathbb{Z}_2
$\pi_1(M_3)$	$[f] = 0$	\mathbb{Z}^3	$\pi_1(\mathcal{B}_1)$
	$[f] \neq 0$:torsion	-	$\pi_1(\mathcal{B}_2)$
	$[f]$:torsionfree	$\Delta(k)$	-

Theorem 5.9 (Halperin-Carlsson conjecture [14]). *Let T^s be an arbitrary effective action on an m -dimensional S^1 -fibred nilBott manifold M of finite type. Then*

$$(5.16) \quad {}_sC_j \leq b_j \quad (= \text{the } j\text{-th Betti number of } M).$$

In particular $2^s \leq \sum_{j=0}^m \text{Rank } H_j(M)$.

Proof. By Corollary 3.4, M admits a homologically injective T^k -action where $k = \text{Rank } C(\pi)$ where $\pi = \pi_1(M)$. Then we have shown in [6] that any homologically injective T^k -actions on any closed aspherical manifold satisfies that

$${}_k C_j \leq b_j.$$

It follows from the result of Conner-Raymond[3] that there is an injective homomorphism $1 \rightarrow \mathbb{Z}^s \rightarrow C(\pi)$. This shows that $s \leq k$ so we obtain

$$(5.17) \quad {}_s C_j \leq b_j \text{ (= the } j\text{-th Betti number of } M).$$

□

Remark 5.10. *This result is obtained when M_i is a real Bott manifold by Masuda, Choi and Oum.*

REFERENCES

- [1] G. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.
- [2] K. Brown, *Cohomology of groups*, GTM, Springer-Verlag, 1982.
- [3] P.E. Conner and F. Raymond, *Actions of compact Lie groups on aspherical manifolds*, *Topology of Manifolds, Proceedings Inst. Univ. of Georgia, Athens, 1969*, Markham (1970), 227-264.
- [4] K. Dekimpe, *Almost-Bieberbach groups: Affine and Polynomial structures*, *Lecture Notes in Math.*, 1639 (1996), Springer-Verlag.
- [5] Y. Kamishima, K.B. Lee and F. Raymond, *The Seifert construction and its applications to infranil manifolds*, *Quart. J. Math., Oxford (2)*, **34** (1983), 433-452.
- [6] Y. Kamishima and M. Nakayam, *On the nil-Bott Tower*, In prepration.
- [7] Y. Kamishima and Admi Nazra, *Seifert fibred structure and rigidity on real Bott towers*, *Contemp. Math.*, vol. 501, 103-122 (2009).
- [8] K.B. Lee and F. Raymond, *Seifert manifold*, *Handbook of Geometric Topology*, R.J. Daverman and R. Sher (eds.), North-Holland (2002), 635-705.
- [9] K.B. Lee and F. Raymond, *Geometric realization of group extensions by the Seifert construction*, *Contemporary Math.* **33** (1984), 353-411.
- [10] S. MacLane, *Homology*, *Die Grundlehren der mathematischen Wissenschaften* vol. 114 Springer, Berlin, New York 1967.
- [11] J. B. Lee, and M. Masuda, *Topology of iterated S^1 -bundles*, Preprint, arXiv:1108.0293 math.AT (2011).
- [12] J. Milnor, *On the 3-dimensional Brieskorn manifolds $M(p, q, r)$* , *Ann. of Math. Studies*, Princeton Univ. Press No. 84 (1975), 175-225.
- [13] M. Nakayam, *Seifert construction for nilpotent groups and application to an S^1 -fibred nil-Bott Tower*.

- [14] V. Puppe, *Multiplicative aspects of the Halperin-Carlsson conjecture*, *Georgian Math. J.* 16 (2009), no. 2, 369-379.
- [15] M.S. Raghunathan, *Discrete subgroups of Lie groups*, *Ergebnisse Math. Grenzgebiete* vol. 68 Springer, Berlin, New York 1972.
- [16] J. Wolf, *Spaces of constant curvature*, *McGraw-Hill, Inc.*, 1967.

DEPARTMENT OF MATHEMATICS AND INFORMATION OF SCIENCES, TOKYO
METROPOLITAN UNIVERSITY, MINAMI-OHSAWA 1-1, HACHIOJI, TOKYO 192-
0397, JAPAN

E-mail address: nakayama-mayumi@ed.tmu.ac.jp